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# On the Clausen integral $\mathrm{Cl}_2(\theta)$ and a related integral

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*Abstract:* In this article, the Clausen integral

$$\mathrm{Cl}_2(\theta) = -\int_0^\theta \ln\left(2\sin\frac{t}{2}\right) \mathrm{d}t$$

whereby $\theta$ is equal to a rational multiple of $\pi$ belonging to $]0,2\pi[$, is developed into a finite series of derivatives of the di-gamma function $\psi$. For the same values of $\theta$, a method is presented to calculate the related integral

$$\int_0^1 \frac{\ln\rho}{\rho^2 - 2\rho\cos\theta + 1}\,\mathrm{d}\rho.$$

*Keywords:* Calculus, Clausen's integral.

## 1. Introduction

We start from a result obtained by Lewin [4, p. 229]

$$\int_0^r \frac{\ln^m(1-x)\ln^n x}{x}\,\mathrm{d}x$$
$$= \int_0^r \log^m(1-\rho e^{i\theta})(\ln\rho + i\theta)^n \frac{\mathrm{d}\rho}{\rho} - i\int_0^\theta \log^m(1-re^{it})(\ln r + it)^n\,\mathrm{d}t, \quad 0 \leqslant r \leqslant 1 \tag{1}$$

in which $m \in \{1, 2,\ldots\}$, $n \in \{0, 1,\ldots\}$, ln means natural logarithm (of a positive real number) and log represents the principal branch of the (Napierian) logarithmic function in the complex plane. When $m = 1$, $n = 0$ and $r = 1$, the equality (1) becomes:

$$\int_0^1 \frac{\ln(1-x)}{x}\,\mathrm{d}x = \int_0^1 \log(1-\rho e^{i\theta})\frac{\mathrm{d}\rho}{\rho} - i\int_0^\theta \log(1-e^{it})\,\mathrm{d}t.$$

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Separating the real and the purely imaginary parts in this equation, we find:

$$\mathrm{Cl}_2(\theta)=\int_0^1 \tan^{-1}\left(\frac{\rho\sin\theta}{1-\rho\cos\theta}\right)\frac{\mathrm{d}\rho}{\rho} \tag{2}$$

and

$$\tfrac{1}{2}\int_0^1 \frac{\ln(\rho^2-2\rho\cos\theta+1)}{\rho}\,\mathrm{d}\rho-\int_0^\theta \tan^{-1}(\cot\tfrac{1}{2}t)\,\mathrm{d}t=-\tfrac{1}{6}\pi^2. \tag{3}$$

Carrying out integration by parts in the right-hand side of (2), we obtain the integral representation of $\mathrm{Cl}_2(\theta)$ on which the calculations in this paper will be based:

$$\mathrm{Cl}_2(\theta)=-\sin\theta\int_0^1 \frac{\ln\rho}{\rho^2-2\rho\cos\theta+1}\,\mathrm{d}\rho. \tag{4}$$

For a few values of $\theta$ in $]0,2\pi[$, the value of $\mathrm{Cl}_2(\theta)$ is known: $\mathrm{Cl}_2(\frac{1}{2}\pi)=G$, whereby $G$ represents Catalan's constant, $\mathrm{Cl}_2(\pi)=0$, and recently, Fettis [2] proved that

$$\mathrm{Cl}_2(\tfrac{1}{3}\pi)=\tfrac{1}{6}\sqrt{3}\left[\psi'(\tfrac{1}{3})-\tfrac{2}{3}\pi^2\right].$$

Our aim is to calculate (4) for the values of $\theta$ announced in the abstract, i.e.,

$$\theta=\frac{p}{q}\pi \tag{5}$$

in which $p$ and $q$ are positive integers satisfying

$$0<p<2q \quad\text{and}\quad (p,q)=1.$$

Essential for our purpose is the fact that $z=\exp(\mathrm{i}p\pi/q)$ is a solution of the binomial equation $z^q=(-1)^p$. Hence, we distinguish successively the cases: $p$ odd and $q$ even; $p$ and $q$ odd; $p$ even and $q$ odd. (The case of $p$ and $q$ even does not occur since we have assumed $(p,q)=1$). Only the case $p$ odd and $q$ even will be dealt with in detail since the treatment of the other two cases runs in a similar way so that it suffices to state the results.

## 2. The case of $p$ odd and $q$ even

*2.1.* In this case, $z=\exp(\mathrm{i}p\pi/q)$ is a solution of $z^q=-1$. The $q$ roots of this equation are represented by

$$z=\exp[\mathrm{i}(2k+1)\pi/q],\quad k=0,1,\dots,q-1,$$

and, taking into account that $q$ is even, it is possible to factorize $\rho^q+1$ as follows:

$$\rho^q+1=(\rho^2-2\rho\cos\theta+1)\cdot P \tag{6}$$

with $\theta$ given by (5) and

$$P=\prod_{\substack{m=0\\ m\neq(p-1)/2\text{ or }(2q-p-1)/2}}^{(q-2)/2}\left(\rho^2-2\rho\cos\frac{2m+1}{q}\pi+1\right).$$

Clearly, $P$ is a polynomial of degree $q-2$ in $\rho$ with coefficients $A_k$ mostly depending on $\theta$, i.e.,

$$P=A_0\rho^{q-2}+A_1\rho^{q-3}+\cdots+A_{q-2}.$$

Inserting this into (6) and identifying the coefficients of equal powers of $\rho$ on both sides, we get:

$$A_0 = 1, \qquad A_1 = 2\cos\theta \tag{7}$$

and the difference-equation type recursion

$$A_{k+1} = (2\cos\theta)A_k - A_{k-1}, \quad k = 1, 2, \ldots, q-3. \tag{7'}$$

The general solution of (7′) is

$$A_k = a\exp(\mathrm{i}k\theta) + b\exp(-\mathrm{i}k\theta), \quad a \in \mathbb{C}, b \in \mathbb{C}.$$

For the particular solution of (7′) which corresponds to the initial conditions (7), there comes:

$$a = \tfrac{1}{2}(1 - \mathrm{i}\cot\theta), \quad b = \tfrac{1}{2}(1 + \mathrm{i}\cot\theta),$$

and so,

$$A_k = \frac{\sin(k+1)\theta}{\sin\theta}, \quad k = 0, 1, \ldots, q-2.$$

Consequently, the development of $(\rho^2 - 2\rho\cos\theta + 1)^{-1}$ with $\theta = p\pi/q$ reads:

$$(\rho^2 - 2\rho\cos\theta + 1)^{-1} = \frac{(\rho^q + 1)^{-1}}{\sin\theta} \sum_{k=0}^{q-2} (\sin(k+1)\theta)\rho^{q-k-2}. \tag{8}$$

*2.2.* We consider the integral

$$I(\alpha) = \int_0^1 \frac{\rho^\alpha}{\rho^2 - 2\rho\cos\theta + 1}\,\mathrm{d}\rho$$

with $\alpha > -1$ and $\theta$ still given by (5). From (8), it follows directly that

$$I(\alpha) = \sum_{k=0}^{q-2} \frac{\sin(k+1)\theta}{\sin\theta} \int_0^1 \frac{\rho^{q+\alpha-k-2}}{\rho^q + 1}\,\mathrm{d}\rho. \tag{9}$$

Applying the substitution $\rho^q = x$, the integral in (9) is transformed into

$$\frac{1}{q}\int_0^1 \frac{x^{(\alpha-k-1)/q}}{x+1}\,\mathrm{d}x$$

and according to Gradshteyn and Ryzhik's Table [3, p. 289 and p. 945], this integral is equal tc

$$\frac{1}{q}\beta\left(1 + \frac{\alpha-k-1}{q}\right) \quad \text{or} \quad \frac{1}{2q}\left[\psi\left(1 + \frac{\alpha-k-1}{2q}\right) - \psi\left(\tfrac{1}{2} + \frac{\alpha-k-1}{2q}\right)\right].$$

Hence, we conclude that

$$I(\alpha) = \frac{1}{2q}\sum_{k=1}^{q-1} \frac{\sin k\theta}{\sin\theta}\left[\psi\left(1 + \frac{\alpha-k}{2q}\right) - \psi\left(\tfrac{1}{2} + \frac{\alpha-k}{2q}\right)\right].$$

Differentiating with respect to $\alpha$, putting $\alpha = 0$ and taking (4) into account, our final result fo the case of $p$ odd and $q$ even is:

$$\mathrm{Cl}_2\left(\frac{p}{q}\pi\right) = -\sin\left(\frac{p}{q}\pi\right)I'(0) = -\frac{1}{4q^2}\sum_{k=1}^{q-1}\left[\psi'\left(1 - \frac{k}{2q}\right) - \psi'\left(\tfrac{1}{2} - \frac{k}{2q}\right)\right]\sin k\frac{p}{q}\pi. \tag{10}$$

*2.3. Examples*

(i) $p=1,\ q=4$.

From (10), it follows directly

$$\mathrm{Cl}_2(\tfrac{1}{4}\pi) = -\tfrac{1}{64}\{\tfrac{1}{2}\sqrt{2}[\psi'(\tfrac{7}{8}) - \psi'(\tfrac{3}{8}) + \psi'(\tfrac{5}{8}) - \psi'(\tfrac{1}{8})] + \psi'(\tfrac{3}{4}) - \psi'(\tfrac{1}{4})\}. \tag{11}$$

Making use of the well-known relations [1, p. 16]

$$\psi'(z) + \psi'(1-z) = \frac{\pi^2}{\sin^2 \pi z}$$

and

$$4\psi'(2z) = \psi'(z) + \psi'(z+\tfrac{1}{2}),$$

this result can be simplified by elimination of various $\psi'$-values. We have:

$$\psi'(\tfrac{7}{8}) = 2\pi^2(2+\sqrt{2}) - \psi'(\tfrac{1}{8}); \quad \psi'(\tfrac{5}{8}) = 2\pi^2(2-\sqrt{2}) - \psi'(\tfrac{3}{8});$$
$$\psi'(\tfrac{1}{4}) = \tfrac{1}{4}\psi'(\tfrac{1}{8}) + \tfrac{1}{4}[2\pi^2(2-\sqrt{2}) - \psi'(\tfrac{3}{8})];$$
$$\psi'(\tfrac{3}{4}) = \tfrac{1}{4}\psi'(\tfrac{3}{8}) + \tfrac{1}{4}[2\pi^2(2+\sqrt{2}) - \psi'(\tfrac{1}{8})].$$

Inserting this into (11) yields

$$\mathrm{Cl}_2(\tfrac{1}{4}\pi) = \tfrac{1}{128}\sqrt{2}[(2-\tfrac{1}{2}\sqrt{2})\psi'(\tfrac{3}{8}) + (2+\tfrac{1}{2}\sqrt{2})\psi'(\tfrac{1}{8}) - 10\pi^2].$$

(ii) $p=1,\ q=6$.

In an analogous way, we find:

$$\mathrm{Cl}_2(\tfrac{1}{6}\pi) = \tfrac{1}{24}[\sqrt{3}\psi'(\tfrac{1}{3}) + 16G - \tfrac{2}{3}\sqrt{3}\pi^2],$$

in which we made use of

$$\psi'(\tfrac{1}{4}) - \psi'(\tfrac{3}{4}) = 16G.$$

## 3. The case of $p$ and $q$ odd, with $q \geqslant 3$

Here the factorization of $\rho^q + 1$ takes the form:

$$\rho^q + 1 = (\rho+1)(\rho^2 - 2\rho\cos\theta + 1)\left(\sum_{k=0}^{q-3} A_k(\theta)\rho^{q-k-3}\right)$$

and it appears that the difference equation connecting the $A$-coefficients is

$$A_{k+1} + (1-2\cos\theta)A_k + (1-2\cos\theta)A_{k-1} + A_{k-2} = 0, \quad k \geqslant 2.$$

The initial conditions read:

$$A_0 = 1, \quad A_1 = 2\cos\theta - 1, \quad A_2 = 2\cos\theta(2\cos\theta - 1).$$

The solution is

$$A_k = \frac{\sin(k+\tfrac{3}{2})\theta + (-1)^k \sin\tfrac{1}{2}\theta}{2\sin\theta\cos\tfrac{1}{2}\theta}, \quad k = 0, 1, \ldots, q-3$$

and in a similar way as before, there comes:

$$\mathrm{Cl}_2(\theta) = -\frac{1}{8q^2}\sum_{k=0}^{q-3}\frac{\sin(k+\frac{3}{2})\theta+(-1)^k\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}$$

$$\times\left[\psi'\left(1-\frac{k+1}{2q}\right)-\psi'\left(\frac{1}{2}-\frac{k+1}{2q}\right)+\psi'\left(1-\frac{k+2}{2q}\right)-\psi'\left(\frac{1}{2}-\frac{k+2}{2q}\right)\right]$$

$$= -\frac{1}{8q^2}\left\{\sum_{k=0}^{q-3}\frac{\sin(k+\frac{3}{2})\theta+(-1)^k\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\left[\psi'\left(1-\frac{k+1}{2q}\right)-\psi'\left(\frac{1}{2}-\frac{k+1}{2q}\right)\right]\right.$$

$$\left.+\sum_{k=1}^{q-2}\frac{\sin(k+\frac{1}{2})\theta-(-1)^k\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\left[\psi'\left(1-\frac{k+1}{2q}\right)-\psi'\left(\frac{1}{2}-\frac{k+1}{2q}\right)\right]\right\}$$

or, finally,

$$\mathrm{Cl}_2\left(\frac{p}{q}\pi\right) = -\frac{1}{4q^2}\sum_{k=1}^{q-1}\left[\psi'\left(1-\frac{k}{2q}\right)-\psi'\left(\frac{1}{2}-\frac{k}{2q}\right)\right]\sin k\frac{p}{q}\pi, \tag{12}$$

the same result as in the case of $p$ odd and $q$ even. For $p=1$, $q=3$, it yields an expression for $\mathrm{Cl}_2(\frac{1}{3}\pi)$ which can be reduced to that of Fettis.

## 4. The case of $p$ even and $q$ odd, with $q \geqslant 3$

In this case, $z=\exp(\mathrm{i}p\pi/q)$ is a solution of the equation $z^q=1$ and the factorization of $\rho^q-1$ becomes

$$\rho^q - 1 = (\rho-1)(\rho^2-2\rho\cos\theta+1)\left(\sum_{k=0}^{q-3}A_k(\theta)\rho^{q-k-3}\right).$$

Solving the corresponding difference equation for the $A$-coefficients, we find:

$$A_k = \frac{\cos\frac{1}{2}\theta-\cos(k+\frac{3}{2})\theta}{2\sin\frac{1}{2}\theta\sin\theta}, \quad k=0,1,\ldots,q-3.$$

After some intermediate calculations, there comes:

$$\mathrm{Cl}_2\left(\frac{p}{q}\pi\right) = -\frac{1}{4q^2}\sum_{k=1}^{q-1}\left[\psi'\left(1-\frac{k}{2q}\right)+\psi'\left(\frac{1}{2}-\frac{k}{2q}\right)\right]\sin k\frac{p}{q}\pi. \tag{13}$$

Applying this to $p=2$, $q=3$, the resulting expression leads to

$$\mathrm{Cl}_2(\tfrac{2}{3}\pi) = \tfrac{1}{9}\sqrt{3}\left[\psi'(\tfrac{1}{3})-\tfrac{2}{3}\pi^2\right]$$

after elimination of $\psi'(\frac{1}{6})$, $\psi'(\frac{2}{3})$ and $\psi'(\frac{5}{6})$. This confirms the known relation [4, p. 104]:

$$\mathrm{Cl}_2(\tfrac{2}{3}\pi) = \tfrac{2}{3}\mathrm{Cl}_2(\tfrac{1}{3}\pi).$$

## 5. Remark

It is now possible to express the tangent integral

$$\mathrm{Ti}_2(x) = \int_0^x \frac{\tan^{-1} t}{t} \, \mathrm{d}t$$

as a finite series in terms of $\psi'$ for certain values of the argument. Indeed, we know a relation between $\mathrm{Ti}_2$ and $\mathrm{Cl}_2$, namely [4, p. 106]:

$$\mathrm{Ti}_2(\tan\theta) = \theta \ln(\tan\theta) + \tfrac{1}{2}\mathrm{Cl}_2(2\theta) + \tfrac{1}{2}\mathrm{Cl}_2(\pi - 2\theta), \quad 0 \leqslant \theta < \tfrac{1}{2}\pi.$$

As an example, we take $\theta = \pi/6$ and conclude that

$$\mathrm{Ti}_2\left(\tfrac{1}{3}\sqrt{3}\right) = -\tfrac{1}{12}\pi \ln 3 + \tfrac{5}{36}\sqrt{3}\left[\psi'\left(\tfrac{1}{3}\right) - \tfrac{2}{3}\pi^2\right].$$

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